# Positive–Definite Generalized Functions and the Heat Equation

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**Summary** : In this note, the correspondence between the solutions of the heat equation and the positive–definite (ultra–) distributions will be considered.

## §0. Introduction.

S.Bochner [1] showed that any positive–definite continuous function can be represented by the Fourier transformation of a finite positive measure. This results was extended by L.Schwartz to the distribution case, [12],[6]. His remarkable result says that any positive–definite distribution must be a tempered one, which is represented by the Fourier transformation of a slowly increasing positive measure.

In this note, we shall investigate the relation between boundary values of the solutions of the heat equation and the positive–definite (ultra–)distributions by using the heat kernel method, [2],[3],[4],[8],[9],[10],[11]. This note contains three theorems. In Theorem 1, we shall show that for any positive–definite continuous function, there corresponds uniquely to a solution of the heat equation satisfying the condition (i),(ii),(iii) in Theorem 1. In Theorem 2, the correspondence between the tempered positive–definite distributions and the solutions of the heat equation satisfying the condition (i),(ii),(iii) in Theorem 2. In Theorem 3, a generalization of the results of Theorem 1 and Theorem 2 to the case of some ultra-distributions(generalized functions) will be considered. To do so, we need an extended Bochner–Schwartz theorem for ultra–distributions which will be proved in Theorem 4.

## §1. Positive–definite continuous functions and Bochner's Theorem

Let $\mathbf{R}^n$ be a n–dimensional Euclidean space whose point is denoted by $x = (x_1, x_2, \cdots, x_n)$. We use the usual notation $\langle x, \xi \rangle = \sum_{j=1}^{n} x_j \xi_j$ and $i = \sqrt{-1}$.

**Definition 1.** Let $f(x)$, $x \in \mathbf{R}^n$, be a (complex–valued) continuous function defined in $\mathbf{R}^n$. We say that a function $f(x)$ is positive–definite if for any finite number of $x^1, x^2, \cdots, x^m \in \mathbf{R}^n$ and $\xi_1, \xi_2, \cdots, \xi_m \in \mathbf{C}$ we have

$$\sum_{j,k=1}^{m} f(x^j - x^k)\xi_j \overline{\xi_k} \geq 0 \tag{1.1}$$

The following facts can be easily shown by the definition.

**Proposition 1.1** Let $f(x)$ be continuous in $\mathbf{R}^n$ and positive-definite. Then we have the following facts :

$$f(0) \geq 0 \tag{1.2}$$

$$|f(x)| \leq f(0), \quad x \in \mathbf{R}^n \tag{1.3}$$

$$f(-x) = \overline{f(x)}, \quad x \in \mathbf{R}^n \tag{1.4}$$

**(Proof)** (1.2) is obtained by setting $m = 1$ in (1.1)

$$f(0)|\xi_1|^2 \geq 0$$

To show (1.4), we set $m = 2$ in (1.1) :

$$f(0)|\xi_1|^2 + f(x^1 - x^2)\xi_1\overline{\xi_2} + f(x^2 - x^1)\xi_2\overline{\xi_1} + f(0)|\xi_2|^2 \geq 0$$

Setting $x^1 = x, x^2 = 0$, we have

$$f(0)|\xi_1|^2 + f(x)\xi_1\overline{\xi_2} + f(-x)\xi_2\overline{\xi_1} + f(0)|\xi_2|^2 \tag{1.5}$$

Since this is real, we take complex conjugate and we have

$$= f(0)|\xi_1|^2 + \overline{f(x)}\,\overline{\xi_1}\xi_2 + \overline{f(-x)}\,\overline{\xi_2}\xi_1 + f(0)|\xi_2|^2$$

From this equality, we have

$$\xi_1\overline{\xi_2}(f(x) - \overline{f(-x)}) + \overline{\xi_1}\xi_2(f(-x) - \overline{f(x)}) = 0 \tag{1.6}$$

Substituting $\xi_1 = 1, \xi_2 = 1$ and setting $A = f(x) - \overline{f(-x)}$, we have

$$A - \bar{A} = 0 \text{ i.e. } A \text{ real}$$

On the other hand, substituting $\xi_1 = i, \xi_2 = 1$ in (1.6), we get

$$iA - i(-A) = 2iA = 0 \text{ i.e. } A = 0$$

Next we shall show (1.3). Since the bilinear form (1.5) is positive–definite,

$$\text{two eigen–values of the matrix } \begin{bmatrix} f(0) & f(x) \\ \overline{f(x)} & f(0) \end{bmatrix} \text{ are } \geq 0$$

This means the roots $\lambda_1, \lambda_2$ of the equation

$$\begin{vmatrix} f(0) - \lambda & f(x) \\ \overline{f(x)} & f(0) - \lambda \end{vmatrix} = \lambda^2 - (f(x) + \overline{f(x)})\lambda + f(0)^2 - |f(x)|^2 = 0$$

are non–negative. So considering the relation of the roots and the coefficients, we have

$$\lambda_1 \lambda_2 = f(0)^2 - |f(x)|^2 \geq 0.$$

(q.e.d)

**Examples of positive–definite functions.**

(a) $f(x) = 1$

(b) $f(x) = e^{iax} (a \in \mathbf{R})$

$$\begin{aligned} \sum_{j,k=1}^{m} f(x^j - x^k)\xi_j \overline{\xi_k} &= \sum_{j,k=1}^{m} e^{ia(x^j - x^k)} \xi_j \overline{\xi_k} \\ &= \sum_{j,k=1}^{m} e^{iax^j} \xi_j \overline{e^{iax^k} \xi_k} = \left|\sum_{j=1}^{m} e^{iax^j} \xi_j\right|^2 \end{aligned}$$

(c) $e^{-ax^2} (a > 0)$

$$e^{-ax^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} \sqrt{\frac{\pi}{a}} e^{-\frac{\xi^2}{4a}} \, d\xi$$

(d) $f(x) = \dfrac{1}{1 \pm ix}$

(e) $f(x) = \dfrac{1}{1 + x^2}$

$$\frac{1}{1+x^2} = \frac{1}{2} \int_{-\infty}^{\infty} e^{ix\xi} e^{-|\xi|} \, d\xi = \frac{1}{2} \frac{1}{ix - 1} + \frac{1}{2} \frac{1}{ix + 1}$$

**Theorem.(Bochner's theorem** [1],[6]) In order that a function $f(x) \in C(\mathbf{R}^n)$ be positive definite, it is nesessary and sufficient that

$$\exists \text{ a positive measure } d\mu(x) \text{ such that } \int_{\mathbf{R}^n} d\mu(\xi) < \infty \text{ and}$$

$$f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x\xi\rangle}\, d\mu(\xi). \tag{1.7}$$

## §2. Relation of positive–definite functions and the heat equation

We denote by $x^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, $x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n$. The n–dimensional heat kernel is given by

$$\begin{aligned} E(x,t) &= (4\pi t)^{-n/2}\, e^{-\frac{x^2}{4t}} \qquad\qquad (t>0) \\ &= (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} e^{-t\xi^2}\, d\xi. \end{aligned}$$

**Theorem 1.** Let $u(x)$ be a continuous positive–definite function in $\mathbf{R}^n$. Then the function $U(x,t) = \int E(x-y,t)u(y)\,dy$ satisfies the following conditions :

(i) $\left(\dfrac{\partial}{\partial t} - \Delta\right)U(x,t) = 0$ in $\mathbf{R}_+^{n+1} = \{(x,t) \in \mathbf{R}^{n+1}, t > 0\}$

(ii) $U(\cdot,t)$ is positive–definite for $\forall t > 0$

(iii) $0 \le U(0,t) \le C = u(0)$.

Conversely, every $C^\infty$–function $U(x,t)$ in $\mathbf{R}_+^{n+1}$ satisfying the conditions (i),(ii),(iii) with a constant $C$ can be expressed in the form $U(x,t) = \int E(x-y,t)u(y)\,dy$ uniquely with $u(x) = U(x,0)$ which is continuous, positive–definite in $\mathbf{R}_+^{n+1}$.

**(Remark.)** We donote the integral in the sense of a pair of a distribution and a test function.

**(Proof.)** ($\Longrightarrow$) By Bochner's theorem there exists a finite positive measure $\mu(\xi)$ in $\mathbf{R}^n$, and $u(x)$ can be represented by

$$u(x) = (2\pi)^{-n} \int e^{ix\xi}\, d\mu(\xi).$$

Substituting this in the expression $U(x,t)$, we get

$$U(x,t) = \int E(x-y,t)\Big((2\pi)^{-n}\int e^{iy\xi}d\mu(\xi)\Big)dy = (\int\Big(2\pi)^{-n}(\int E(x-y,t)e^{iy\xi}\,dy\Big)d\mu(\xi)$$

$$= (2\pi)^{-n}\int e^{ix\xi}\Big(\int E(x-y,t)e^{iy\xi}dy\Big)d\mu(\xi) = (2\pi)^{-n}\int e^{ix\xi}e^{-t\xi^2}d\mu(\xi)$$

This implies positive definiteness of $U(x,t)$ for any $t>0$.

As $U(x,t)$ becomes positive–definite, by (1.3), we have

$$|U(x,t)| \leq U(0,t) \leq \int E(y,t)|u(y)|\,dy \leq u(0) \equiv C$$

($\Longleftarrow$) Conversely, let $U(x,t)$ satisfies (i),(ii) and (iii) with some constant $C>0$. Then by §1, (1.2),(1.3), we obtain

$$|U(x,t)| \leq U(0,t) \leq C \qquad (x,t) \in \mathbf{R}^{n+1}_{+}$$

Furthermore, by Theorem 19.2 in [10] or Theorem 5.7 in [11], there exists uniquely

$$u = U(x,0) \in \boldsymbol{\mathcal{S}}'(\mathbf{R}^n).$$

and we have the expression $U(x,t) = \int E(x-y,t)u(y)\,dy$. Using the Fourier transform, we have

$$\widehat{U}(\xi,t) = e^{-t\xi^2}\widehat{u}(\xi).$$

By Bochner's theorem, there exists a positive finite measure $\mu_t(\xi)$ such that

$$\widehat{U}(\xi,t) = \mu_t(\xi) = e^{-t\xi^2}\widehat{u}(\xi) \geq 0.$$

This means $\widehat{u}$ must be a positive measure.

On the other hand, we have

$$U(x,t) = (2\pi)^{-n}\int e^{i\langle x,\xi\rangle}\,e^{-t\xi^2}\widehat{u}(\xi)\,d\xi. \tag{2.1}$$

By (iii)

$$U(0,t) = (2\pi)^{-n}\int e^{-t\xi^2}\,\widehat{u}(\xi)\,d\xi \leq C$$

By using Fatou's lemma and tending $t \downarrow 0$, we have $(2\pi)^{-n}\int \widehat{u}(\xi)\,d\xi \leq C$, which means that $\widehat{u}(\xi)$ is a finite measure. By using Lebesgue's convergence theorem in (2.1) and tending $t \to 0$, we have

$$u = U(x,0) = (2\pi)^{-n}\int e^{i\langle x,\xi\rangle}\widehat{u}(\xi)\,d\xi.$$

This shows $u$ is continuous and positive–definite. (q.e.d.)

Now we shall consider the relation of the positive–definite distributions $u \in \mathcal{S}'(\mathbf{R}^n)$ and the solutions of the heat equation.

**Definition 2.** $u \in \mathcal{S}'(\mathbf{R}^n)$ is said to be positive–definite if and only if

$$\langle u, \varphi * \varphi^* \rangle \geq 0, \qquad \forall \varphi \in \mathcal{S}(\mathbf{R}^n), \quad \varphi^*(x) = \overline{\varphi(-x)}.$$

We shall describe Bochner–Schwartz theorem and Riesz–Kakutani's theorem. The former is the extension of Bochner's theorem to the case $\mathcal{S}'$. The latter is to certificate the existence of a positive measure.

**Theorem.(Bochner–Schwartz theorem [6],[12])** In order that a distribution $f(x) \in \mathcal{S}'(\mathbf{R}^n)$ be positive–definite, it is nesessary and sufficient that

$\exists$ a positive measure $d\mu(x)$ and $N \geq 0$ such that $\int_{\mathbf{R}^n} (1+|\xi|^2)^{-N} \, d\mu(\xi) < \infty$ and

$$f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i(x,\xi)} \, d\mu(\xi) \tag{2.2}$$

**Theorem.(Riesz–Kakutani's theorem [3])** Every continuous, positive linear functional on $C_0(\mathbf{R}^n)$ is given by

$$\langle F, \varphi \rangle = \int \varphi(x) \, d\mu(x),$$

where $\mu$ is some positive measure (not necessarily finite).

**Theorem 2.** Let $u(x)$ be a distribution $\in \mathcal{S}'(\mathbf{R}^n)$ and positive–definite. Then the function $U(x,t) = \langle E(x-\cdot,t), u(\cdot) \rangle = \int E(x-y,t)u(y)\,dy$
satisfies the following conditions :

(i) $\left(\dfrac{\partial}{\partial t} - \Delta\right) U(x,t) = 0$ in $\mathbf{R}^{n+1}_+$

(ii) $U(\cdot,t)$ is positive–definite for $\forall t > 0$

(iii) $0 \leq U(0,t) \leq Ct^{-N} \quad (\exists N > 0) \qquad 0 < t < \infty$

Conversely, every $C^\infty$–function $U(x,t)$ in $\mathbf{R}^{n+1}_+$ satisfing (i),(ii),(iii) can be expressed in the form $U(x,t) = \int E(x-y,t)u(y)\,dy$ uniquely with $u(x) = U(x,0)$ which is $\in \mathcal{S}'$ and positive–definite.

**(Proof)** $(\Longrightarrow)$ See the proof of Theorem 1,[10],[11].

$(\Longleftarrow)$ If $U(x,t)$ satisfies (ii) and (iii), then by §1, (1.2),(1.3), we have

$$|U(x,t)| \leq U(0,t) \leq C(1+t^{-N}) \qquad (x,t) \in \mathbf{R}_+^{n+1}$$

Hence by Theorem 19.2,[10] or Theorem 5.7,[11], there exists a unique

$$u = U(x,0) \in \boldsymbol{\mathcal{S}}'(\mathbf{R}^n)$$

and we have the representation $U(x,t) = \int E(x-y,t)u(y)\,dy$, and

$$0 \leq \int U(x,t)\varphi * \varphi^*(x)\,dx, \qquad \forall \varphi \in \boldsymbol{\mathcal{S}}(\mathbf{R}^n). \tag{2.3}$$

As $t \downarrow 0$, we have

$$\langle u(x), \varphi * \varphi^* \rangle \geq 0.$$

Substituting the integral representation of $U(x,t)$ in (2.3), then we can get

$$\int \left( \int E(x-y,t)u(y)\,dy \right) \varphi * \varphi^*(x)\,dx.$$

Changing the order of the integrals, we have

$$= \int \left( \int E(x-y,t)\varphi * \varphi^*(x)\,dx \right) u(y)\,dy$$

Using the representation of $U(x,t)$, we have

$$= \int U(x,t)\varphi * \varphi^*(x)\,dx$$

Using Parseval's equality, we have

$$\int e^{-t\xi^2}\widehat{u}(\xi)|\widehat{\varphi}|^2\,d\xi \geq 0.$$

By Bochner–Schwartz theorem, there exists a finite measure $\mu_t(\xi)$ and

$$\widehat{U}(\xi,t) = \mu_t(\xi) = e^{-t\xi^2}\widehat{u}(\xi) \geq 0$$

Tending $t \downarrow 0$, we have $\langle \widehat{u}(\xi), |\varphi(\xi)|^2 \rangle \geq 0$. This means that $\widehat{u}$ is multiplicatively positive in $\boldsymbol{\mathcal{S}}$. We know every multiplicatively positive distribution in $\boldsymbol{\mathcal{S}}'$ is a positive one by the argument given in §2,Chapter 2 in [6]. Hence, by Riesz–Kakutani's theorem, $\widehat{u}$ is a positive measure.

We have to show $\hat{u}$ is a tempered measure, that is to say, there is a positive constant $k$ such that

$$\int (1+|\xi|^2)^{-k}\, \hat{u}\, d\xi < \infty$$

Since $\hat{u}$ is continuous in $\mathcal{S}'(\mathbf{R}^n)$, we have the following inequality

$$|\langle \hat{u}, \varphi\rangle| \le C \sum_{|\alpha|,|\beta|\le k} \sup |\xi^\alpha \partial_\xi^\beta \varphi(\xi)|, \quad \varphi \in \mathcal{S}(\mathbf{R}^n) \tag{2.4}$$

Taking $\varphi(\xi) = (1+|\xi|^2)^{-k}$, we set $U_\varphi(\xi,t) = \int E(\xi-\eta,t)\varphi(\eta)\, d\eta = \varphi_t(\xi)$, which plays a role of a barrier function. We substitute $\varphi_t(\xi)$ in the right–hand–side of (2.4). We have

$$\xi^\alpha \partial_\xi^\beta \varphi_t(\xi) = \xi^\alpha \int \partial_\xi^\beta E(\xi-\eta,t)\varphi_t(\eta)\, d\eta$$

Considering $\partial_\xi^\beta E(\xi-\eta,t) = (-\partial_\eta)^\beta E(\xi-\eta,t)$, integrating by parts and using the inequality $|\xi^\alpha| \le 2^{|\alpha|}(|\xi-\eta|^{|\alpha|} + |\eta|^{|\alpha|})$, we get the terms of the right–hand–side in (2.4) with $\varphi = \varphi_t$ are finite. Hence we have

$$|\langle \hat{u}, \varphi_t\rangle| \le C \qquad \text{for } (0<t<T).$$

Tending $t \downarrow 0$, we have

$$\int (1+|\xi|^2)^{-k}\, \hat{u}\, d\xi < \infty$$

(q.e.d.)

The next theorem is concerned with the ultra-distributions, that is, generalized functions in $(\mathcal{S}_r^s)'$(in the sense of Gelfand–Shilov).

We shall give the folloing definition.

**Definition 4.** ([5]) We say that a function $\varphi(x)$ is $\in \mathcal{S}_{r,A}^{s,B}(\mathbf{R}^n)$ if there exist $0<r,s,\ 1\le r+s\le\infty$ and $C$ such that

$$|x^\alpha D_x^\beta \varphi(x)| \le C A^{|\alpha|} B^{|\beta|} \alpha!^r \beta!^s \text{ for } \forall\ \alpha,\beta \in \mathbf{N}^n$$

holds. We denote by $\mathcal{S}_r^s(\mathbf{R}^n)$ the inductive limit of $\mathcal{S}_{r,A}^{s,B}(\mathbf{R}^n)$ as $A, B \to \infty$. And we denote by $(\mathcal{S}_r^s(\mathbf{R}^n))'$ the set of the generalized functions on $\mathcal{S}_r^s(\mathbf{R}^n)$.

**Definition 5.** $u \in (\mathcal{S}_r^s(\mathbf{R}^n))'$ is said to be positive–definite if and only if

$$\langle u, \varphi * \varphi^*\rangle \ge 0, \qquad \forall \varphi \in \mathcal{S}_r^s(\mathbf{R}^n), \quad \varphi^*(x) = \overline{\varphi(-x)}.$$

Then the following theorem holds.

**Theorem 3** We assume that $\frac{1}{2} \leq r, s < \infty$. Let $u(x)$ be a generalized function $\in (\boldsymbol{S}_r^s(\mathbf{R}^n))'$ and positive–definite. Then the function
$U(x,t) = \langle E(x-y,t), u(y)\rangle = \int E(x-y,t)u(y)\,dy$ satisfies the following conditions :

(i) $\left(\dfrac{\partial}{\partial t} - \Delta\right)U(x,t) = 0$ in $\mathbf{R}_+^{n+1}$.

(ii) $U(\cdot,t)$ is positive–definite for $\forall t > 0$.

(iii) In case $\frac{1}{2} < s < \infty$, for $\forall\, \epsilon > 0, \forall T > 0$ we have $0 \leq U(0,t) \leq C_\epsilon\, e^{\epsilon t^{\frac{-1}{2s-1}}}$ $\quad 0 < t < T$,

where $C_\epsilon$ is a constant depending on $\epsilon$.

(iii)$'$ In case $s = \frac{1}{2}$, for $\forall T > 0$ we have $0 \leq U(0,t) \leq C(t) < \infty, \qquad 0 < t < T$,

where $C(t)$ is a constant depending on $t$.

Conversely, every $C^\infty$–function $U(x,t)$ in $\mathbf{R}_+^{n+1}$, satisfing (i),(ii),(iii) or (i),(ii),(iii)$'$ can be expressed in the form $U(x,t) = \int E(x-y,t)u(y)\,dy$ uniquely with $u(x) = U(x,0)$ which is $\in (\boldsymbol{S}_r^s(\mathbf{R}^n))'$ and positive–definite .

**Remark** (1) In §3, we shall show that $\hat{u}$ is a positive measure and for $\forall\, \epsilon > 0$

$$\int \hat{u}(\xi)e^{-\epsilon|\xi|^{\frac{1}{s}}}\,d\xi < \infty,$$

i.e. infra–exponentially increasing.

(2) In case $s = 1$ in Theorem 3, we have $|U(x,t)| \leq U(0,t) \leq C_\epsilon e^{\frac{\epsilon}{t}}$ so that $u \in \boldsymbol{\mathcal{B}}(\mathbf{R}^n)$, Fourier hyperfunction.

**(Proof)** ($\Longrightarrow$) By the extended Bochner–Schwartz theorem(Theorem 4 in §3), there exists a (infra–exponential) positive measure $\mu(\xi)$ such that

$$u(x) = (2\pi)^{-n}\int e^{\langle x,\xi\rangle}d\mu(\xi).$$

Since $E(\cdot,t) \in \boldsymbol{S}_{1/2}^{1/2}$, $u \in (\boldsymbol{S}_r^s)'$, $\hat{u} \in (\boldsymbol{S}_s^r)'$, we have

$$U(x,t) = \int E(x-y,t)u(y)\,dy = (2\pi)^{-n}\int e^{i\langle x,\xi\rangle}e^{-t\xi^2}\hat{u}(\xi)\,d\xi \in C^\infty(\mathbf{R}_+^{n+1})$$

and satisfes (ii).

For (iii), we have to estimate the integral

$$U(0,t) = (2\pi)^{-n} \int e^{-t\xi^2} \widehat{u}(\xi)\, d\xi = (2\pi)^{-n} \sup_{\xi} e^{-t\xi^2 + \epsilon|\xi|^{1/s}} \int e^{-\epsilon|\xi|^{1/s}} \widehat{u}(\xi)\, d\xi.$$

We have the inequality

$$0 \le U(0,t) \le C_\epsilon \sup_{\xi} e^{-t\xi^2 + \epsilon|\xi|^{1/s}}$$

by setting $C_\epsilon = (2\pi)^{-n} \int e^{-\epsilon|\xi|^{1/s}} \widehat{u}(\xi)\, d\xi$. Estimating the sup and setting $-\epsilon^{\frac{2s}{2s-1}} 2^{\frac{2s}{1-2s}} (1-2s)$ by $\epsilon$, we have

$$U(0,t) \le C_\epsilon e^{\epsilon t^{-1/(2s-1)}}$$

To prove (iii)$'$, we estimate the integral for $t > \epsilon$

$$U(0,t) = (2\pi)^{n} \int e^{t\xi^2} \widehat{u}(\xi)\, d\xi = (2\pi)^{n} \sup_{\xi} e^{-t\xi^2 + \epsilon|\xi|^2} \int e^{-\epsilon|\xi|^2} \widehat{u}(\xi)\, d\xi.$$

For $t > \epsilon$, sup is estimated by $\le 1$ and the integral is estimated by $C_\epsilon$. Hence we obtain (iii)$'$.

($\Longleftarrow$) In case (iii)

$$|U(x,t)| \le U(0,t) \le C_\epsilon e^{\epsilon t^{\frac{-1}{2s-1}}}, \quad 0 < t < T.$$

Using Theorem 2.1 in [2], for $\frac{1}{2} \le \forall r < \infty$, we have uniquely

$$u = U(x,0) \in (\boldsymbol{S}^s_r(\mathbf{R}^n))'.$$

Furthermore we can represent

$$U(x,t) = \langle E(x-y,t), u(y) \rangle = \int E(x-y,t) u(y)\, dy.$$

By the assumption, we have

$$\int U(x,t) \varphi * \varphi^*\, dx \ge 0 \quad \forall \varphi \in \boldsymbol{S}^s_r(\mathbf{R}^n). \tag{2.5}$$

Tending $t \to 0$, we get

$$\langle u, \varphi * \varphi^* \rangle \ge 0$$

Substituting the integral representation of $U(x,t)$ in (2.5), then we can get

$$\int \langle E(x-y,t), u(y) \rangle\, \varphi * \varphi^*(x)\, dx.$$

By continuity of the generalized function and the definition of the integral, we have

$$= \langle \int E(x-y,t)\, \varphi * \varphi^*(x)\, dx, u(y)\rangle$$

Using Parseval's equality, we have

$$\int e^{-t\xi^2} \widehat{u}(\xi) |\widehat{\varphi}|^2\, d\xi \geq 0.$$

By the extended Bochner–Schwartz theorem(Theorem 4), there exists a positive measure $\mu_t(\xi)$ and

$$\widehat{U}(\xi,t) = \mu_t(\xi) = e^{-t\xi^2} \widehat{u}(\xi) \geq 0$$

Tending $t \downarrow 0$, we have $\langle \widehat{u}(\xi), |\varphi(\xi)|^2\rangle \geq 0$. This means that $\widehat{u}$ is multiplicatively positive in $\mathcal{S}_s^r$. We can see that every multiplicatively positive generalized function in $(\mathcal{S}_s^r(\mathbf{R}^n))'$ is a positive one by almost the same argument given in §2,Chapter 2 in [6]. Hence, by Riesz–Kakutani's theorem, $\widehat{u}$ is a positive measure. By Theorem 4, we have

$$\int e^{-\epsilon|\xi|^{1/s}} \widehat{u}(\xi)\, d\xi < \infty$$

(q.e.d)

## §3. Extended Bochner–Schwartz theorem

We shall show the extended Bochner–Schwartz theorem for the generalized functions in $(\mathcal{S}_r^s(\mathbf{R}^n))'$.

**Theorem 4.** In order that a generalised function $u \in (\mathcal{S}_r^s(\mathbf{R}^n))'$ be positive–definite, it is nesessary and sufficient that there exists a positive measure $d\mu(\xi)$ such that for any $\epsilon \geq 0$ we have $\int_{\mathbf{R}^n} e^{-\epsilon|\xi|^{1/s}}\, d\mu(\xi) < \infty$ and

$$u(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle}\, d\mu(\xi). \tag{3.1}$$

**(Proof)** ($\Longleftarrow$) The sufficiency of the proof can be obtained by almost the same way as in the proof of Theorem 1 and 2, where the heat kernel method might be used effectively.

($\Longrightarrow$) The proof is divided into 4 steps.

(Step 1) $(\widetilde{\mathcal{S}_r^s}) = \mathcal{S}_s^r$ by Gelfand–Shirov [5]. Since $\hat{u}^*$, for $\forall\, \varphi \in \mathcal{S}_s^r$, we have

$$0 \leq \langle u, \varphi * \varphi^*\rangle = \langle \hat{\hat{u}}^*, \varphi * \varphi^*\rangle = \langle \widehat{u}^*, \widehat{\varphi * \varphi^*}\rangle = \langle \widehat{u}^*, |\widehat{\varphi}|^2\rangle.$$

So $\widehat{u}^*$ is a multiplicatively positive in $\boldsymbol{S}^r_s$. Hence we have $\widehat{u}^*$ is positive in $\boldsymbol{S}^r_s$, then in $C_0$ by using the heat kernel method.

(Step 2) By Riesz–Kakutani's theorem, $\widehat{u}^*(\xi)$ is a positive measure.

(Step 3) Applying the Theorem 4.2 in Chung–Kim [4] to non–negative solution of the heat equation $U^*(\xi,t) = \int E(\xi-\eta,t)\widehat{u}^*(\eta)\,d\eta \geq 0$, we have

$$0 \leq U^*(\xi,t) \leq t^{-n/2}e^{\epsilon|\xi|^{1/s}}, \quad 0 < t \leq T.$$

(Step 4) Since the growth order of $U(\xi,t)$ in $t$ is $t^{-n/2}$, we have

$$0 \leq U^*(\xi,0) = \widehat{u}^*(\xi) \in \mathcal{D}'(\mathbf{R}^n).$$

Setting $m = \left[\frac{n}{2}\right] + 1$.

$$f(t) = \begin{cases} \dfrac{t^{m-1}}{(m-1)!} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0. \end{cases}$$

For $f(t)$, $v(t)$ and $w(t)$ are constructed satisfying following conditions :

$$v(t) = f(t) \text{ for } t \leq 1, \text{supp}(v) \subset [0,2],$$

$$(d/dt)^m v(t) = \delta(t) + w(t),\ \text{supp}(w) \subset [1,2]. \tag{3.2}$$

By the Theorem 19.2 in [10] or Theorem 5.7 in [11], we have

$$0 \leq \widetilde{U}^*(\xi,t) = \int_0^2 U^*(\xi, q+t)v(s)\,dq \in O(e^{\epsilon|\xi|^{1/s}}).$$

$\widetilde{U}^*(\xi,t)$ is $C^\infty$ in $\mathbf{R}^n \times (0,2)$ and

$$|\widetilde{U}^*(\xi,t)| \leq C\exp(\epsilon|\xi|^{1/s})$$

We can use $\widetilde{U}^*(\xi,t)$ is continuously extended to $\mathbf{R}^n \times [0,2)$.

$$\left(\frac{\partial}{\partial t} - \Delta\right)\widetilde{U}^*(\xi,t) = 0 \text{ in } \mathbf{R}^n \times (0,2). \tag{3.3}$$

Integrating by part and using (3.2) we have the equality

$$(-\Delta)^m\widetilde{U}^*(\xi,t) = \left(-\frac{d}{dt}\right)^m \widetilde{U}^*(\xi,t) = U^*(\xi,t) + \int_0^2 U^*(\xi,t+q)w(q)\,dq$$

We set $h(\xi,t) = \int_0^2 U^*(\xi,t+q)w(q)\,dq$. We see $h(\xi,t)$ is $C^\infty$ in $\mathbf{R}^n \times (0,2)$ which is continuously extended to $\mathbf{R}^n \times [0,2)$. Furthermore we see

$$|h(\xi,t)| \leq C\exp(\epsilon|\xi|^{1/s}).$$

Setting $g(\xi) = \widetilde{U}^*(\xi,0)$ and tending $t \to 0$, we have

$$(-\Delta)^m g(\xi) = U^*(\xi,0) + h(\xi,0).$$

This means

$$\langle(-\Delta)^m \widetilde{U}^*(\xi,t), \varphi(\xi)\rangle = \langle U^*, \varphi(\xi)\rangle + \langle h(\xi,t), \varphi(\xi)\rangle.$$

Left–hand–side of the above equality is equal to

$$\langle \widetilde{U}^*(\xi,t), (-\Delta)^m \varphi(\xi)\rangle$$

Tending $t \to 0$, we have

$$\langle g, (-\Delta)^m \varphi(\xi)\rangle = \langle u^*, \varphi\rangle + \langle h(\xi), \varphi\rangle.$$

So we obtain the estimate (3.1)

$$0 \leq \int e^{-\epsilon|\xi|^{1/s}}\, \widehat{u}(\xi)\, d\xi < \infty$$

(q.e.d.)

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