On some Remarks of Representation Theorem of Sobolev's Spaces and Boundary Value Problems

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Summary : In this note, the Sobolev space are analysed, then it is applied to the boundary problems of some differential operators.

1 Introduction

The theory of distributions constructed systematically by Schwartz [4] has many applications to several domains in mathematics. In particular, in the theory of partial differential equations and the probability theory, it plays the fundamental role. Rozanov's result [3] was intended to apply it to the probability theory. One of his results is to determine the structure of the functional space $(H^m(\Omega))'$. Then, using this result, he discussed the existence of the solutions of the differential equations appearing in the probability theory. His method is somewhat different from the method used in the theory of partial differential equations. Therefore we summerize here the results using the usual notations and the usual results in Mizohata [2]. Our result might be used to the problem of positive–definite distributions elsewhere(cf. [1]).

This note contains 8 sections. Section 2 is devoted to the notations and the main results in a book of Rozanov [3]. In §3, we prove Theorem 3.1 using the result in Mizohata [2]. In §4 we descrive the representation theorem in the case of $(H^m(\Omega))'$. In the latter half of the paper we shall give some applications of the representation theorem. §5 is devoted to the construction of the Riemann function in the case of the ordinary differential equations. In §6 we shall give an example to Theorem 4.2 in the case of ordinary differential equations. In §7, we apply Theorem 4.2 to the case of elliptic partial differential operators. In §8 we shall apply again Theorem 4.2 to the problem of heat equation.

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2 Notations and a representation of Sobolev's spaces

Let $\mathbf{R}^n$ be n–dimensional Euclidean space with its point $x = (x_1, x_2, \cdots, x_n)$. Let $\Omega \subset \mathbf{R}^n$ be an open set. The followings are the usual notations

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n},\ \alpha = (\alpha_1, \alpha_2 \cdots, \alpha_n),\ |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

where $D_j = -i\partial/\partial x_j$. Furthermore $C_0^\infty(\Omega)$ denotes the set of infinetely differentiable functions with compact support in Ω.

Definition 1. We define by $\mathcal{D}(\Omega)$ the set of $C_0^\infty(\Omega)$ which has the topology in the following sense *i.e.* $\{\varphi_n(x)\} \to 0$ means $\operatorname{supp} \varphi_n \subset K, n = 1, 2, \cdots$, for some compact set $K \subset \Omega$, and for any non–negative integer m

$$|\varphi_n|_{m,K} = \max_{|\alpha| \leq m, K} |D^\alpha \varphi_n| \to 0 \qquad \text{as } n \to \infty$$

Definition 2. We denote by $\mathcal{D}'(\Omega)$ the set of continuous linear functionals on $\mathcal{D}(\Omega)$.

Definition 3. We define by $\mathcal{E}'(\Omega)$ the set of continuous linear functionals on $\mathcal{D}(\Omega)$ with compact support in Ω.

Definition 4. We donote by $\mathcal{S} = \mathcal{S}\ (\mathbf{R}^n)$ the set of $C^\infty(\mathbf{R}^n)$ which has the topology in the folloing sense *i.e.* $\{\varphi_n(x)\} \to 0$ means that

$$\sup_{x \in \mathbf{R}^n, |\alpha| \leq m, |\beta| \leq k} |x^\alpha D^\beta \varphi_n| \to 0 \text{ as } n \to \infty \qquad \forall\ m, k$$

Definition 5. We denote by $\mathcal{S}' = \mathcal{S}'(\mathbf{R}^n)$ the set of continuous linear functionals on $\mathcal{S}(\mathbf{R}^n)$.

For $\varphi \in \mathcal{S}$ we define the Fourier transform by

$$\mathcal{F}\varphi(\xi) = \widehat{\varphi}(\xi) = \int e^{-i\langle x, \xi \rangle} \varphi(x)\, dx$$

where $\langle x, \xi \rangle = x_1\xi_1 + \cdots + x_n\xi_n$. We define the Fourier transform of $T \in \mathcal{S}'$ by

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle \qquad \forall \varphi \in \mathcal{S}'$$

For $f \in \mathcal{D}(\Omega)$, we denote by $\|f\|_p$ a norm

$$\|f\|_p^2 = \int |\hat{f}(\xi)|^2 (1 + |\xi|^2)^p\, d\xi$$

for any real number p. For positive p, W denotes the completion of $\mathcal{D}(\mathbf{R}^n)$ with respect to the norm $\|f\|_p$. We also denote by $W(\Omega) = [\mathcal{D}(\Omega)]$ the closure of $\mathcal{D}(\Omega)$ with this norm $\|f\|_p$. X denotes the completion of $\mathcal{D}(\mathbf{R}^n)$ with respect to the norm $\|f\|_{-p}$. We also denote by $X(\Omega) = [\mathcal{D}(\Omega)]$ the closure of $\mathcal{D}(\Omega)$ with this norm $\|f\|_{-p}$. It is easy to see $W^* = X$.

Theorem 2.1. $X(\Omega) = [\mathcal{D}(\Omega)]$ *consists of* $f \in X,\ \operatorname{supp} f \subset \overline{\Omega}$.

The proof is in [3]. We prove it, in §3, by another method in the case of the Sobolev space.

Let $\mathcal{P}$ be an elliptic positive operator of order $2m$ satisfying

$$\|\varphi\|_W^2 = \langle \varphi, \mathcal{P}\varphi \rangle \asymp \|\varphi\|_m^2, \quad \varphi \in \mathcal{D}(\mathbf{R}^n), \tag{2.1}$$

where $\asymp$ denotes the equivalent norm. We denote $X = \overset{\circ}{W}{}_2^{-m}(\Omega)$ then we have

$$X = \mathcal{P}W. \tag{2.2}$$

Denote by $\partial\Omega = \Gamma$ and

$$X(\Gamma) = \{f \in X(\Omega), \quad \operatorname{supp} f \subset \Gamma\}. \tag{2.3}$$

Then we have the following theorem.

Thoeorem 2.2. (representation theorem)

$$X(\Omega) = [\mathcal{P}\mathcal{D}(\Omega)] \oplus X(\Gamma) \tag{2.4}$$

The proof is given also in [3]. More precise representation formula will be given in §4.

We define the following non–isotropic norm which will be used in §8 for treating the heat equation. For $u \in \mathcal{D}(\mathbf{R}^{n+1})$, we denote

$$\|u\|_{(m,s)}^2 = \int_{\mathbf{R}^{n+1}} (1 + |\xi|^{2m} + |\eta|^{2s})|\hat{u}(\xi,\eta)|^2 \, d\xi d\eta$$

, where $x \in \mathbf{R}^n, y \in \mathbf{R}$ where $\hat{u}(\xi,\eta)$ denote the Fourier transform of $u(x,y)$. We denote

$$H^{(m,s)}(\mathbf{R}^{n+1}) = W_2^{(m,s)}(\mathbf{R}^{n+1}) = W$$

and its dual space with respect to the L^2–inner product by

$$H^{-(m,s)}(\mathbf{R}^{n+1}) = W_2^{-(m,s)}(\mathbf{R}^{n+1}) = X.$$

Furthermore, the restrictions to Ω of the spaces $H^{(m,s)}(\mathbf{R}^{n+1})$ and $H^{-(m,s)}(\mathbf{R}^{n+1})$ are denoted by $H^{(m,s)}(\Omega)$ and $H^{-(m,s)}(\Omega)$, respectively.

3 Some properties of $(\overset{\circ}{H}{}^m(\Omega))'$

In the following m is a fixed positive integer. Furthermore we denote $\partial_j = \partial/\partial x_j$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

Theorem 3.1. (representation theorem) *If $u \in (\overset{\circ}{H}{}^m(\Omega))'$, then there exists $\{f_\alpha\} \in L^2(\Omega)$ and*

$$u = \sum_{|\alpha| \le m} \partial^\alpha f_\alpha \tag{3.1}$$

These $\{f_\alpha\}$ are not necessarily unique.

Remark. We denote by $(\overset{\circ}{H}{}^m(\Omega))' = \overset{\circ}{H}{}^{-m}(\Omega)$

(Proof) For $u \in (\overset{\circ}{H}{}^m(\Omega))'$, $\langle u, \varphi \rangle, \varphi \in \overset{\circ}{H}{}^m(\Omega)$, is a bounded linear functional on $\overset{\circ}{H}{}^m(\Omega)$. Using the Riesz theorem(cf. [2,Theorem 2.10,p.73]), there exists $g \in H^m(\Omega)$ such that

$$\begin{aligned}\langle u, \varphi \rangle &= (\varphi, g)_{m,L^2} = \sum_{|\alpha| \le m} (\partial^\alpha \varphi, \partial^\alpha g)_{L^2} \\ &= \sum_{|\alpha| \le m} \langle (-1)^{|\alpha|} \partial^\alpha \partial^\alpha g, \varphi \rangle\end{aligned}$$

Hence, by setting $f_\alpha = (-1)^{|\alpha|} \partial^\alpha g$, we have

$$u = \sum_{|\alpha| \le m} \partial^\alpha f_\alpha.$$

■

Theorem 3.2. *$\mathcal{D}(\Omega)$ is dense in $(\overset{\circ}{H}{}^m(\Omega))'$.*

(Proof) By Theorem 3.1, $u \in (\overset{\circ}{H}{}^m(\Omega))'$ is represented by $u = \sum_{|\alpha| \le m} \partial^\alpha f_\alpha$, where $\exists \{f_\alpha\}$, $f_\alpha \in L^2(\Omega)$. For one of the above f_α's, there is an approximate sequence $\{\varphi_{\alpha j}\}_{j=1}^\infty \subset \mathcal{D}(\Omega)$ which satisfies $\|\varphi_{\alpha j} - f_\alpha\|_{L^2(\Omega)} \to 0$(cf. [2,Proposition 2.4,p.67]). We put $u_j = \sum_{|\alpha| \le m} \partial^\alpha \varphi_{\alpha j}$, then $u_j \in \mathcal{D}$ and $u_j \to u$ in $(\overset{\circ}{H}{}^m(\Omega))'$. Hence $u \in (\overset{\circ}{H}{}^m(\Omega))'$.

For $\forall \varphi \in \overset{\circ}{H}{}^m(\Omega)$, we have

$$\begin{aligned}|\langle u, \varphi \rangle| &= \left| \sum_{|\alpha| \le m} \langle \partial^\alpha f_\alpha, \varphi \rangle \right| \\ &\le \sum_{|\alpha| \le m} \|f_\alpha\|_{L^2} \cdot \|\partial^\alpha \varphi\|_{L^2}.\end{aligned}$$

By the Cauchy–Schwarz inequality, we get

$$\leq \Big(\sum_{|\alpha|\leq m} \|f_\alpha\|_{L^2}^2\Big)^{1/2}\Big(\|\sum_{|\alpha|\leq m} \|\partial^\alpha\varphi\|_{L^2}^2\Big)^{1/2}$$

$$\leq \Big(\sum_{|\alpha|\leq m} \|f_\alpha\|_{L^2}^2\Big)^{1/2} \|\varphi\|_{H^m}$$

Hence we obtain

$$\|u_j - u\|_{H^{-m}(\Omega)}^2 = \sum \|f_\alpha - \varphi_{\alpha j}\|_{L^2}^2 \to 0,$$

which is to be proved. ■

4 The representation of $(H^m(\Omega))'$

In this section we shall determine the representation of the distribution $\mathcal{E}'(\Omega)$ which has a support in one point and also the representation of the distribution $(H^m(\Omega))'$ with its support touched to the boundary $\Gamma = \partial\Omega$. This is the precision of the Theorem 2.2 in §2. In the following, we denote by δ the Dirac delta function and also by $C(U)$ the continuous function on U.

Theorem 4.1. *Let $f \in \mathcal{E}'$ and $\operatorname{supp} f = \{O\}$. Then we have, for $\exists m$ non-negative integer,*

$$f = \sum_{|\alpha|\leq m} C_\alpha \partial^\alpha \delta \tag{4.1}$$

(Proof) The following proof is due to Yoshida–Ito [6,p.135].

Since $f \in \mathcal{E}'$, f is represented by $f = \sum_{|\alpha|\leq m} \partial^\alpha g_\alpha$ where $g_\alpha \in C(U)$ and U is a neiborhood of the origin O. Let $\varphi \in \mathcal{E}$ satisfying $D^\alpha\varphi(0) = 0$ for $|\alpha| \leq m$. Then we have

$$\langle f, \varphi\rangle = 0$$

from the following argument.

Let $\psi \in \mathcal{D}$ and satisfy

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \leq 1/2 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

For a fixed φ we put $\varphi_j(x) = \psi(jx)\varphi(x)$. Then for $|\alpha| = m$ we have $\sup_{|x|\leq 1/j} |\partial^\alpha\varphi(x)| \to 0$. Further for $|\alpha| < m$ we get

$$\begin{aligned} \partial^\alpha\varphi(x) &= \int_0^1 \frac{d}{dt}(\partial^\alpha\varphi(tx))dt \\ &= \sum_{i=1}^n x_i \int_0^1 \frac{d}{dx_i}(\partial^\alpha\varphi(tx))dt. \end{aligned}$$

So we see by the induction on $|\alpha|$

$$\sup_{|x|\leq 1/j} |\partial^\alpha \varphi(x)| = o(j^{|\alpha|-m}).$$

By the Leibniz formula we have

$$\partial^\alpha \varphi_j(x) = \sum_{\alpha\geq\beta} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\varphi(x) j^{|\beta|}(\partial^\beta \psi)(jx).$$

So it obtains

$$\begin{aligned}\sup_{x\in U} |\partial^\alpha \varphi_j(x)| &= \sup_{|x|\leq 1/j} |\partial^\alpha \varphi_j(x)| \\ &= \sup_{x,|\alpha|\leq m} |(\partial^\alpha \psi)(x)| \sum_{\alpha\geq\beta} \binom{\alpha}{\beta} j^{|\beta|} \sup |\partial^{\alpha-\beta}\varphi(x)| = o(j^{|\alpha|-m})\end{aligned}$$

Since φ_j and its derivatives up to the order m converge uniformly, we get

$$\lim_{j\to\infty} \langle f, \varphi_j \rangle \to 0.$$

On the other–hand, since $\varphi_j(x) = \varphi(x)$ in $|x| \leq 1/j$ and the support of f is the origin, $\langle f, \varphi \rangle = \langle f, \varphi_j \rangle$. It follows that $\langle f, \varphi \rangle = 0$.

For $\forall \varphi \in \mathcal{E}$, we put

$$r_m = \varphi(x) - \sum_{|\alpha|\leq m} \frac{x^\alpha}{\alpha!} \partial^\alpha \varphi(0) (\in \mathcal{E}).$$

As $\partial^\alpha r_m(0) = 0(|\alpha| \leq m)$, we get $\langle f, r_m \rangle = 0$. Puting $C_\alpha = \dfrac{(-1)^{|\alpha|}}{\alpha!} \langle f, x^\alpha \rangle$, it follows that

$$f = \sum_{|\alpha|\leq m} C_\alpha \partial^\alpha \delta$$

■

In the case of $\Omega = [0, \infty)$, we obtain the representation theorem for $T \in (H^m(\Omega))'$ by the same argument as above. For n–dimensional case, we divide the domain Ω into the patches which are two types. The first ones are in the interior of Ω and the others are touched to the boundary. The former are corresponding to the open neighborhood of the origin, and the latter are corresponding to the neiborhood of $x_n \geq 0$ and its boundaries are contained in $x_n = 0$.

Theorem 4.2. *Let T be in $((H^m(\Omega))'$. Then there exists $\{g_\alpha\} \subset L^2$ and $x_k \in H^{-(m-k-1/2)}(\Gamma)$ $(k = 0, \cdots, m-1)$ such that*

$$T = \sum_{|\alpha|\leq m} \partial^\alpha g_\alpha + \sum_{k=0}^{m-1} x_k \otimes \delta^{(k)}(\Gamma) \tag{4.2}$$

Remark. We denote $(H^m(\Omega))' = H^{-m}(\Omega)$.

(Proof) Since $T \in ((H^m(\Omega))'$, $\langle T, \varphi\rangle$ is a linear functional on $\varphi \in \overset{\circ}{H}{}^m(\Omega)$. We have

$$|\langle T, \varphi\rangle| \le C(\sum_{|\alpha|\le m} \|\partial^\alpha \varphi\|^2_{L^2(\Omega)})^{1/2} = C\|\varphi\|_{m,L^2}$$

By Theorem 3.1, there is a set $\{g_\alpha\} \subset L^2$ such that

$$T_1 = \sum_{|\alpha|\le m} \partial^\alpha g_\alpha, \quad \langle T_1, \varphi\rangle = \langle T, \varphi\rangle$$

Setting $T_2 = T - T_1$, since $\langle T, \varphi\rangle = \langle T_1, \varphi\rangle$, we get $\operatorname{supp} T_2 \subset \Gamma$. Fixing $x^{(0)} = (x_1^{(0)}, \cdots, x_{n-1}^{(0)}, 0) = (x^{(0)\prime}, 0)$ and U is a neighborhood of O. We continue the same argument as in one variable. We put

$$\varphi(x) = \varphi(x', 0) + \partial_n \varphi(x', 0) x_n + \cdots + \partial_n^m \varphi(x', 0) \frac{x_n^m}{m!} + \psi(x)$$

where $\psi(x) = O(x_n^{m+1})$. Hence $\langle T_2, \varphi\rangle = 0$. we consider the particular case $T_2 = T_2' \otimes T_2''$.

$$\begin{aligned}\langle T_2, \partial_n^k \varphi(x', 0)\frac{x^k}{k!}\rangle &= \langle T_2' \otimes T_2'', \partial_n^k \varphi(x', 0)\frac{x^k}{k!}\rangle \\ &= \langle T_2' \otimes \delta^{(k)}, \varphi\rangle\end{aligned}$$

By $T_2 = T - T_1$, it follows that

$$T = T_1 + T_2 = \sum_{|\alpha|\le m} \partial^\alpha g_\alpha + \sum_{k=0}^{m-1} x_k \otimes \delta^{(k)}(\Gamma)$$

This completes the proof. ■

5 Ordinary differential equations and the Riemann function

This section is due to Yosida [5,p.53]. This is the preparation of the representation of the solution of the ordinary differential equation which will be given in §6.

We consider the ordinary differential equation of the folloing type

$$\mathcal{L}y = y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_n(x) y = q(x) \tag{5.1}$$

where $p_1(x), \cdots, p_n(x)$ and the right–hand–side $q(x)$ are continuous in the interval D.

Theorem 5.1. *For any point x_1 in the interval D and any data $\eta, \eta', \cdots, \eta^{(n-1)}$, there exists a unique continuous solution $y(x)$ in D satisfying the equation (5.1) and the initial values*

$$y(x_1) = \eta, y'(x_1) = \eta', \cdots, y^{(n-1)}(x_1) = \eta^{(n-1)} \tag{5.2}$$

Theorem 5.2. *The difference $z(x) = y_1(x) - y_2(x)$ of the two solutions of the equation (5.1) satisfies the equation*

$$z^{(n)} + p_1(x)z^{(n-1)} + \cdots + p_n(x)z = 0. \tag{5.3}$$

Hence any solution of the equation (5.1) is written by the sum of a particular solution of (5.1) and a solution of the homogeneous equation (5.3).

Theorem 5.3. *For a system of fundamental solutions $\{z_j(x)\}$ of (5.3), we take the unknown functions $\{u_j(x)\}_{j=1}^n$ which satisfy*

$$(\mathrm{E}) \quad \left\{ \begin{array}{lllll} z_1u_1' & +z_2u_2' & +\cdots & +z_nu_n' & =0 \\ z_1'u_1' & +z_2'u_2' & +\cdots & +z_n'u_n' & =0 \\ \cdots & \cdots & +\cdots & +\cdots & \cdots \\ z_1^{(n-1)}u_1' & +z_2^{(n-1)}u_2' & +\cdots & +z_n^{(n-1)}u_n' & =q(x) \end{array} \right. \tag{5.4}$$

Then we have the solution of (5.1) by setting

$$y(x) = \sum_{i=1}^{n} z_i(x)u_i(x).$$

(Proof) Differentiating (5.4) succesively, we have by using (5.3)

$$\begin{array}{ll} y(x) & = \sum_{i=1}^{n} z_i(x)u_i(x) \\ y'(x) & = \sum_{i=1}^{n} z_i'(x)u_i(x) \\ \cdots & \cdots \\ y^{(n-1)}(x) & = \sum_{i=1}^{n} z_i^{(n-1)}(x)u_i(x) \\ y^{(n)}(x) & = \sum_{i=1}^{n} z_i^{(n)}(x)u_i(x) + q(x) \end{array}$$

Therefore considering $\{z_j(x)\}_{j=1}^n$ solutions of (5.3), we see that $y(x)$ satisfies (5.1). ∎

Theorem 5.4. *The above method obtaining a particular solution of (5.1), is the same as the folloing method. The variable x is in an open interval $a \leq x \leq b$. We take continuous functions $\{a_i(x)\}_{i=1}^n$ and choose continuous ones $\{b_i(x)\}_{i=1}^n$ satisfying the folloing equations*

$$\left\{ \begin{array}{llll} (b_1(x) - a_1(x))z_1(x) & +\cdots & +(b_n(x) - a_n(x))z_n(x) & =0 \\ (b_1(x) - a_1(x))z_1'(x) & +\cdots & +(b_n(x) - a_n(x))z_n'(x) & =0 \\ \cdots & \cdots & \cdots & \cdots \\ (b_1(x) - a_1(x))z_1^{(n-2)}(x) & +\cdots & +(b_n(x) - a_n(x))z_n^{(n-2)}(x) & =0 \\ (b_1(x) - a_1(x))z_1^{(n-1)}(x) & +\cdots & +(b_n(x) - a_n(x))z_n^{(n-1)}(x) & =q(x) \end{array} \right.$$

Then we define

$$R(x,\xi) = \begin{cases} \sum_{j=1}^{n} a_j(x) z_j(\xi) & \text{if } a \le x \le \xi \\ \sum_{j=1}^{n} b_j(x) z_j(\xi) & \text{if } \xi < x \le b \end{cases}$$

Then we have a particular solution of (5.1) as the following form

$$y(x) = \int_a^b R(x,\xi) q(\xi)\, d\xi$$

(Proof) Since $(b_j(x) - a_j(x))q(x)$ is coinside with the solution $u_j'(x)$ of (5.4), we have

$$\begin{aligned} \int_a^b R(x,\xi)q(\xi)\,d\xi &= \sum_{j=1}^{n} z_j(x)\Big\{\int_a^x b_j(\xi)q(\xi)\,d\xi + \int_x^b a_j(\xi)q(\xi)\,d\xi\Big\} \\ &= \sum_{j=1}^{n} z_j(x) \int_a^x u_j'(\xi)\,d\xi + \sum_{j=1}^{n} z_j(x)\int_a^b a_j(\xi)q(\xi)\,d\xi \\ &= \sum_{j=1}^{n} z_j(x)(u_j(x) - u_j(a)) + \sum_{j=1}^{n} z_j(x)\int_a^b a_j(\xi)q(\xi)\,d\xi \\ &= \sum_{j=1}^{n} z_j(x)u_j(x) + \sum_{j=1}^{n} c_j z_j(x) \end{aligned}$$

Hence $y(x)$ is represented by the sum of a particular solution and the solution of (5.3). ■

Remark. This function $R(x,\xi)$ is called the Riemann function of the equation (5.3). Here the choise of a_j is not unique, so the function is not uniquely defined.

Example 5.1 We consider the Cauchy problem for $\mathcal{L}\varphi = \psi$ with the boundary deta $\varphi^{(j)}(0) = 0 (j = 0, \cdots, 2m-1)$ in the interval $I = [0,\infty)$. Then we have, by Theorem 5.4,

$$\varphi(t) = \int_0^t g(t,s)\psi(s)\,ds = (g_t^-, \psi), \quad t > 0. \tag{5.5}$$

Here we set the Riemann function $R(x,\xi)$ in the above as $g(t,s)$ and we put

$$g_t^-(t,s) = \begin{cases} g(t,s) & \text{if } t \ge s \\ 0 & \text{if } t < s. \end{cases}$$

By Theorem 3.1, we have $X = (\overset{\circ}{H}{}^{2m}(I))' = \overset{\circ}{H}{}^{-2m}(I)$. Then $x = \mathcal{L}^* g \in \mathcal{L}^* L_2(I)$ and

$$\langle x, u\rangle = \langle \mathcal{L}^* g_t^-, u\rangle = \langle g_t^-, \mathcal{L}u\rangle = \langle g_t^-, f\rangle.$$

For a resolvent representation of the solution (5.5), we have

$$\varphi^{(k)}(t) = (\delta_t^{(k)}, \varphi) = (g_t^{(k)}, \mathcal{L}\varphi) = (\mathcal{L}^* g_t^{(k)}, \varphi), \qquad \varphi \in \mathcal{D}$$

for $k = 0, 1, \cdots, 2m-1$. So $X = \mathcal{L}^* L_2(I)$ contains the $\delta_t^{(k)}$ functions.

Example 5.2 We consider the Cauchy problem for $\mathcal{L}u = f$ with the boundary deta $u^{(j)}(0) = \xi_j (j = 0, \cdots, 2m-1)$ in the interval $[0, \infty)$. By (5.5) we have

$$u(t) = (g_t^-, f) + \sum_{j=0}^{2m-1} u_j(t)\xi_j$$

where $\{u_j(t)\}$ is a system of fundamental solutions of $\mathcal{L}u = 0$ with the initial data $u_j^{(k)} = \delta_{jk}$. For $x \in X$, we have $x = \mathcal{L}^* g_t^- + \sum_{j=0}^{2m-1} x_j \delta^{(j)}$. Therefore it follows that

$$\begin{aligned}\langle x, u\rangle &= \langle \mathcal{L}^* g_t^-, u\rangle + \textstyle\sum_{j=0}^{2m-1}\langle x_j\delta^{(j)}, u\rangle \\ &= \langle g_t^-, \mathcal{L}u\rangle + \textstyle\sum_{j=0}^{2m-1}\langle x_j, u^{(j)}(0)\rangle \\ &= \langle g_t^-, f\rangle + \textstyle\sum_{j=0}^{2m-1} x_j\xi_j\end{aligned}$$

We get

$$u_t = (\delta_t, u) = (g_t^-, \mathcal{L}u) + \sum_{j=0}^{2m-1} \xi_j u_j(t) = (\mathcal{L}^* g_t^-, u) + \sum_{j=0}^{2m-1} \xi_j u_j(t)$$

that is, we have

$$\delta_t = \mathcal{L}^* g_t^- + \sum_{j=0}^{2m-1} \delta^{(j)} u_j(t).$$

This shows Theorem 4.2 with $g_t^- \in L^2$.

6 The case of ordinary differential equations

Let $\mathcal{L}$ be an ordinary differential operator of order $2m$ with constant coefficients :

$$\mathcal{L} = \sum_{k=0}^{2m} a_k\left(\frac{d^k}{dx^k}\right)$$

and I is an interval $[a, b]$.

Theorem 6.1. *The boundary value problem*

$$(\mathrm{P}) \quad \begin{cases} \mathcal{L}u &= f \quad \text{in } I \\ u^{(j)}(a) &= g_0^{(j)} \quad (j = 0, 1, \cdots, m-1) \\ u^{(j)}(b) &= g_1^{(j)} \quad (j = 0, 1, \cdots, m-1) \end{cases} \tag{6.1}$$

is given. If it has a unique solution, then for any $f \in L^2(I)$ *and any data* $\{g_0^{(j)}, g_1^{(j)}\}_{0 \le j \le 2m-1}$, *there exists a solution* $u \in H^{2m}(I)$ *satisfying (P).*

(Proof) For $\forall \xi \in (\overset{\circ}{H}{}^{2m}(I))'$, by Theorem 4.2, there exists $\eta \in L^2(I)$

$$\xi = \mathcal{L}^* \eta + \sum_{j=0}^{n} \xi_0^{(j)} \delta_a^{(j)} + \sum_{j=0}^{n} \xi_1^{(j)} \delta_b^{(j)}, \tag{6.2}$$

where

$$\|\xi\|_{-2m} \asymp \|\eta\|_{L^2} + \sum_{j=0}^{m-1} |\xi_0^{(j)}| + \sum_{j=0}^{m-1} |\xi_1^{(j)}|. \tag{6.3}$$

We put

$$F(\xi) = \langle \eta, f \rangle + \sum_{j=0}^{m-1} \xi_0^{(j)} g_0^{(j)} + \sum_{j=0}^{m-1} \xi_1^{(j)} g_1^{(j)}.$$

Then we obtain

$$\begin{aligned} |F(\xi)| &\le |\langle \eta, f \rangle| + \sum_{j=0}^{m-1} |\xi_0^{(j)}|\,|g_0^{(j)}| + \sum_{j=0}^{m-1} |\xi_1^{(j)}|\,|g_1^{(j)}| \\ &\le \|\eta\|\,\|f\| + \sum_{j=0}^{m-1} |\xi_0^{(j)}|\,|g_0^{(j)}| + \sum_{j=0}^{m-1} |\xi_1^{(j)}|\,|g_1^{(j)}| \\ &\le C\Big(\|\eta\| + \sum_{j=0}^{m-1} |\xi_0^{(j)}| + \sum_{j=0}^{m-1} |\xi_1^{(j)}|\Big). \end{aligned}$$

By (6.3), we then have

$$|F(\xi)| \le C\|\xi\|_{-2m}.$$

So $F(\xi)$ is a bounded linear functional on $(H^{2m}(I))'$. Therefore there is a $u \in H^{2m}(I)$ satisfying

$$F(\xi) = \langle \xi, u \rangle$$

For $\varphi \in \mathcal{D}$, we put $\xi = \mathcal{L}^* \varphi$. Since $\mathcal{L}^* \varphi$ is 0 on the boundary ∂I, we get

$$\langle \mathcal{L}^* \varphi, u \rangle = \langle \varphi, f \rangle \quad i.e. \quad \langle \varphi, \mathcal{L} u \rangle = \langle \varphi, f \rangle$$

This implies $\mathcal{L} u = f$. Putting $u = u_0^{(j)} \otimes \delta_a^{(j)}$ we have $u_0^{(j)} = g_0^{(j)}$. By the same substitution, we also have $u_1^{(j)} = g_1^{(j)}$. These facts show that u is a solution of the boundary problem (P). ■

7 The case of elliptic partial differential equation

Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary Γ. Furthermore, let $\mathcal{P}$ be an elliptic partial differential operator of order $2m$ and positive–definite :

$$\mathcal{P} = \sum_{|k|\leq 2m} a_k \partial^k > 0 \tag{7.1}$$

with

$$\|\varphi\|_{H^m} = (\varphi, \mathcal{P}\varphi)^{1/2} \asymp \|\varphi\|_m, \quad \varphi \in \mathcal{D}(\Omega). \tag{7.2}$$

For $\forall \xi \in H^m(\Omega)$, $\eta = \mathcal{P}\xi \in H^{-m}(\Omega)$ and

$$\eta = \langle \varphi, \eta \rangle = (\mathcal{P}\varphi, \xi), \qquad \varphi \in \mathcal{D}(\Omega). \tag{7.3}$$

Furthermore the folloing relation is satisfied

$$\|\varphi\|_m \asymp \|\mathcal{P}\varphi\|_{-m}. \tag{7.4}$$

We consider the boundary value problem of this operator :

$$(\mathrm{P}) \qquad \begin{cases} \mathcal{P}\xi & = \eta \quad \text{in} \quad \Omega \\ \xi^{(j)}(a) & = \xi_k \quad \text{on } \Gamma, \ (j = 0, 1, \cdots, m-1) \end{cases} \tag{7.5}$$

where $\xi_k \in H^{m-k-1/2}(\Gamma)$.

Theorem 7.1. *For any $\eta \in (H^m(\Omega))'$, there exists an unique solution $\xi \in H^m(\Omega)$ of the boundary value problem (7.5).*

(Proof) By Theorem 4.2, we have

$$X = (H^m(\Omega))' = \mathcal{P}H^m(\Omega) \oplus \sum_{k=0}^{m-1} H^{-(m-k-1/2)}(\Gamma)$$

where $\mathcal{P}H^m(\Omega) = [\mathcal{P}\mathcal{D}(\Omega)]$. Hence, for any $x \in (H^m(\Omega))'$, we have

$$x = \mathcal{P}u + \sum_{j=0}^{m-1} x_k \otimes \delta^{(k)}. \tag{7.6}$$

Here $u \in H^m(\Omega)$, $x_k \in H^{-(m-k-1/2)}(\Gamma)(k = 0, 1, \cdots, m-1)$ and

$$\|x\|_{-m} \asymp \|u\|_m + \sum_{k=0}^{m-1} \|x_k\|_{-(m-k-1/2)}.$$

For any $x \in (H^m(\Omega))'$, we define

$$F(x) = \langle u, \eta \rangle + \sum_{k=0}^{m-1} \langle x_k, \xi_k \rangle.$$

This indicates $F(x)$ is a bounded linear functional on $(H^m(\Omega))'$. Because the following estimates holds for $x \in (H^m(\Omega))'$

$$\begin{aligned} |F(x)| &\le |\langle u, \eta\rangle| + \sum_{k=0}^{m-1} |\langle x_k, \xi_k\rangle| \\ &\le C\Big(\|u\|_m + \sum_{k=0}^{m-1} \|x_k\|_{-(m-k-1/2)}\Big) \le C\|x\|_{-m}. \end{aligned}$$

Hence there exists $\xi \in H^m(\Omega)$ satisfying

$$\begin{aligned} \langle \mathcal{P}\varphi, \xi\rangle &= \langle \varphi, \eta\rangle & &\forall \varphi \in \mathcal{D}(\Omega) \\ \langle x_k \otimes \delta^{(k)}, \xi\rangle &= \langle x_k, \xi^{(k)}\rangle = \langle x_k, \xi_k\rangle & &x_k \in H^{-(m-k-1/2)}(\Gamma), \\ & & &(k = 0, \cdots, m-1). \end{aligned}$$

This implies that ξ satisfies (7.5). ■

8 The case of heat equation

In this section, we consider the heat equation

$$\frac{\partial u}{\partial t} = \Delta u + f \tag{8.1}$$

in $\mathbf{R}^n \times \mathbf{R}$. We put

$$\mathcal{L} = \partial_t - \Delta$$

and

$$\mathcal{P} = \mathcal{L}^*\mathcal{L} = -\partial_t^2 + \Delta^2.$$

The following notations are given in §2, that is,

$$W = [\mathcal{D}] = \overset{\circ}{W}{}^{(2,1)}(\mathbf{R}^{n+1}),$$

$$\|\varphi\|_W = \langle \varphi, \mathcal{P}\varphi\rangle^{1/2} \asymp \|\varphi\|_{(2,1)}, \qquad \varphi \in \mathcal{D}.$$

Then we also have

$$X = W^* = W^{-(2,1)}(\mathbf{R}^{n+1}),$$

$$X = \mathcal{L}^* L_2(\mathbf{R}^{n+1}).$$

We denote by $\langle \cdot, \cdot\rangle$ the inner product of $L^2(\mathbf{R}^{n+1})$ and by $(\cdot, \cdot)$ the inner product of $L^2(\mathbf{R}^n)$. Hence we have the existence and the uniqueness theorem as follows.

Theorem 8.1. *For $f \in L_2(\mathbf{R}^{n+1})$, there exists an unique solution $u \in W_2^{(2,1)}(\mathbf{R}^{n+1})$ of (8.1).*

(Proof)

$$g = g' \otimes 1_{(a,b)}, \quad g' \in \mathcal{D}(\mathbf{R}^n).$$

The set of functions of this type is dense in $L^2(\mathbf{R}^{n+1})$. Denote by $u = u(t, \cdot) = u|_t$. We have

$$\begin{aligned} x = \mathcal{L}^* g &= (-\partial_t - \Delta)g \\ &= -g' \otimes (\partial_t 1_{(a,b)}) - (\Delta g') \otimes 1_{(a,b)} \\ &= -g' \otimes (\delta_a - \delta_b) - (\Delta g') \otimes 1_{(a,b)} \end{aligned}$$

This shows Theorem 4.2. Hence taking the inner product with u, we have the following equality

$$\begin{aligned} \langle x, u \rangle = \langle \mathcal{L}^* g, u \rangle &= \langle g' \otimes (\delta_b - \delta_a), u \rangle - \langle \Delta g' \otimes 1_{(a,b)}, u \rangle \\ &= (g', u|_b) - (g', u|_a) - \langle g' \oplus 1_{(a,b)}, \Delta u \rangle \\ &= \int_a^b (g' \otimes 1_{(a,b)}, \partial_s u|_s) ds - \langle g' \otimes 1_{(a,b)}, \Delta u \rangle \\ &= \langle g' \otimes 1_{(a,b)}, \partial_t u \rangle - \langle g' \otimes 1_{(a,b)}, \Delta u \rangle \\ &= \langle x' \otimes 1_{(a,b)}, \partial_t u - \Delta u \rangle = \langle g' \otimes 1_{(a,b)}, f \rangle \end{aligned}$$

From this we have

$$\langle x, u \rangle = \langle g, f \rangle.$$

By the same arguments as in §6,§7, it follows that $\langle g, f \rangle$ is a linear functional on $x \in X$. Consequently we have an unique solution $u \in W$. ∎

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