A survey of credibilistic processes

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1 Summary

Stochastic process in probability theory is an important tool to describe dynamical phenomena under uncertainty. However, in the real world, we often encounter the difficult problem which could not be treated with by using only the probability theory. To deal with such complex problem, Liu [5, 6] proposed the credibility theory which is a mathematical fuzzy model. In this paper, referring to Kageyama and Iwamura [3], we introduce a construction of credibilistic processes.

2 Credibility theory

Recall some definitions and basic lemmas on credibility space [3, 5–7].

For any non-empty set X, we denote by $\mathscr{P}(X)$ the power set of X. Let Θ be an arbitrary non-empty set. If the following four axioms are satisfied, the set function Cr is said to be a credibility measure on $\mathscr{P}(\Theta)$.

Axiom 1. $\mathrm{Cr}\{\Theta\} = 1$.

Axiom 2. Cr *is increasing, i.e.,* $\mathrm{Cr}\{A\} \leq \mathrm{Cr}\{B\}$ *whenever* $A \subset B$.

Axiom 3. Cr *is self-dual, i.e.,* $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$ *for any* $A \in \mathscr{P}(\Theta)$.

Axiom 4. $\mathrm{Cr}\{\cup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ *for any* $\{A_i\} \subset \mathscr{P}(\Theta)$ *with* $\sup_i \mathrm{Cr}\{A_i\} < 0.5$.

Then the triplet $(\Theta, \mathscr{P}(\Theta), \mathrm{Cr}\{\cdot\})$ is called a credibility space.

Let $\mathfrak{R}$ be the set of real numbers. For any non-empty set X, a function $p : X \to [0,1]$ is said to satisfy Condition $\mathscr{K}$ with X if the following (1a) and (1b) hold:

$$\sup_{x \in X} p(x) \geq 0.5, \tag{1a}$$

$$p(x^*) + \sup_{x \neq x^*} p(x) = 1 \ \ if \ \ p(x^*) \geq 0.5. \tag{1b}$$

Lemma 2.1 *(Credibility extension theorem [7]) For any non-empty set X, let p be any function satisfying Condition $\mathscr{K}$ with X. Then, a set function* $\mathrm{Cr}\{\cdot\}$ *defined by the following (2) and (3) becomes a credibility measure on* $\mathscr{P}(X)$*:*

$$\mathrm{Cr}\{A\} = \begin{cases} p(A) & if \ \ p(A) < 0.5, \\ 1 - p(A^c) & if \ \ p(A) \geq 0.5, \end{cases} \quad (A \in \mathscr{P}(X)) \tag{2}$$

where for $D \in \mathscr{P}(X)$

$$p(D) = \sup_{x \in D} p(x). \tag{3}$$

A function from a credibility space $(\Theta, \mathscr{P}(\Theta), \mathrm{Cr}\{\cdot\})$ to $\mathfrak{R}$ is called a fuzzy variable.

Definition 2.2 *A function* $p : \mathfrak{R} \to [0,1]$ *is called a prescriptive function of a fuzzy variable* ξ *if it satisfies that for any* $A \in \mathscr{P}(\mathfrak{R})$,

$$\mathrm{Cr}\{\xi \in A\} = \begin{cases} p(A) & if \ \ p(A) < 0.5, \\ 1 - p(A^c) & if \ \ p(A) \geq 0.5, \end{cases} \tag{4}$$

where $p(D)$ *is defined by (3)* $(D \in \mathscr{P}(\mathfrak{R}))$.

A family of fuzzy variables is called a credibilistic process, say $\{\xi_t : t = 0, 1, \cdots\}$. For any non-empty set X, the set of all functions $p : X \to [0, 0.5]$ satisfying Condition $\mathscr{K}$ with X will be denoted by $\mathscr{K}(X)$ emphasizing the domain X. For any non-empty sets X and Y, a credibilistic kernel on Y given X is a function $q(\cdot|\cdot)$ on $Y \times X$ which satisfies that $q(\cdot|x) \in \mathscr{K}(Y)$ for all $x \in X$. The set of all credibilistic kernels on Y given X will be denoted by $\mathscr{K}(Y|X)$. We need the following Lemma to construct a credibilistic process in Section 3.

Lemma 2.3 ([3]) *Let X, Y be any non-empty sets. Let* $q \in \mathscr{K}(Y|X)$*. For any* $p \in \mathscr{K}(X)$*, we define a function* $g : X \times Y \to [0, 0.5]$ *by*

$$g(x,y) = p(x) \wedge q(y|x) \ for \ x \in X, \ y \in Y. \tag{5}$$

Then g satisfies Condition $\mathscr{K}$ with $X \times Y$*, that is,* $g \in \mathscr{K}(X \times Y)$.

3 Construction of credibilistic processes

For a sequence of non-empty subset of $\mathfrak{R}$, $\{X_t : t = 1, 2, \cdots, T\}$, we denote by $\mathscr{H}_t$ the Cartesian product of $\{X_n : n = 1, 2, \cdots, t\}$, that is, $\mathscr{H}_t = X_1 \times X_2 \times \cdots \times X_t (1 \leq t \leq T)$. Let $\{q_t : t = 2, 3, \cdots, T\}$ be a family of credibilistic kernels with $q_{t+1} \in \mathscr{K}(X_{t+1}|\mathscr{H}_t)$ for all $t (1 \leq t \leq T-1)$.

Lemma 3.1 ([3]) *Let* $p_1 \in \mathscr{K}(X_1)$*. For each* $t (1 \leq t \leq T)$*, we define a function* $g_t : \mathscr{H}_t \to [0, 0.5]$ *by*

$$\begin{aligned} g_t(x_1, x_2, \cdots, x_t) = p_1(x_1) \wedge q_2(x_2|x_1) \wedge q_3(x_3|x_1, x_2) \wedge \\ \cdots \wedge q_t(x_t|x_1, x_2, \cdots, x_{t-1}) \end{aligned} \tag{6}$$

for all $(x_1, \cdots, x_t) \in \mathscr{H}_t$.
Then, it holds that $g_t \in \mathscr{K}(\mathscr{H}_t)$ *for* $t (1 \leq t \leq T)$.

Proof. This follows from Lemma 2.3 immediately. ■
The following Theorem is our main result.

Theorem 3.2 ([3]) *Let* $(\mathscr{H}_T, \mathscr{P}(\mathscr{H}_T), \mathrm{Cr}\{\cdot\})$ *be a credibility space, where* $\mathrm{Cr}\{\cdot\}$ *is constructed by constituents of* $p_1 \in \mathscr{K}(X_1)$ *and* $q_{t+1} \in \mathscr{K}(X_{t+1}|\mathscr{H}_t)(1 \le t \le T-1)$. *For each* $t(1 \le t \le T)$, *we define a fuzzy variable* $\xi_t : \mathscr{H}_T \to X_t$ *by*

$$\xi_t(x_1, x_2, \cdots, x_T) = x_t \tag{7}$$

for any $(x_1, x_2, \cdots, x_T) \in \mathscr{H}_T$. *Then, it holds that* $\{\xi_t : t = 1, 2, \cdots, T\}$ *is a credibilistic process and the prescriptive function of* ξ_t, $p_t : X_t \to [0, 0.5]$, *is given by*

$$p_t(x_t) = \sup_{(x_1, x_2, \cdots, x_{t-1}) \in \mathscr{H}_{t-1}} g_t(x_1, x_2, \cdots, x_{t-1}, x_t). \tag{8}$$

Proof. For any $A \in \mathscr{P}(X_t)$, we observe that p_t defined in (8) satisfies (4) with $\xi = \xi_t$ and $p(x) = p_t(x)$ for $x \in X_t$, which implies that p_t is a prescriptive function of ξ_t. ■

Now, let us consider the infinite case.

Theorem 3.3 ([3]) *Let* $(\mathscr{H}_\infty, \mathscr{P}(\mathscr{H}_\infty), \mathrm{Cr}\{\cdot\})$ *be a credibility space constructed by constituents of* $p_1 \in \mathscr{K}(X_1)$ *and* $q_{t+1} \in \mathscr{K}(X_{t+1}|\mathscr{H}_t)(1 \le t < \infty)$. *For each* $t(1 \le t < \infty)$, *we define a fuzzy variable* $\xi_t : \mathscr{H}_\infty \to X_t$ *by*

$$\xi_t(x) = x_t \; for \; any \; x = (x_1, x_2, \cdots) \in \mathscr{H}_\infty,$$

where $\mathscr{H}_\infty := X_1 \times X_2 \times \cdots$. *Then,* $\{\xi_t : t = 1, 2, \cdots\}$ *is a credibilistic process and* p_t *defined in (8) is a prescriptive function of* ξ_t.

Proof. Trivial. ■

4 Future work

In dealing with decision making under uncertainty, the chain rule for stochastic process is not enough. The credibility theory, initiated by Liu, is defined on the basis of the axiomatization , which provides an important technique for analysis of complex problem (cf. [1, 2]). In this paper, the credibilistic process based on the axiomatization of credibility measures was proposed from a family of credibilistic kernels applying the credibility extension theorems. Also, we studied hybrid process which is a more flexible mathematical model(cf. [4]). And furthermore, Liu [8] proposed the uncertain measure M;

Axiom 1. $M\{\Gamma\} = 1$ *for the universal set* Γ.

Axiom 2. $M\{\Lambda\} + M\{\Lambda^c\} = 1$ *for any event* Λ.

Axiom 3. *For every countable sequence of event* $\Lambda_1, \Lambda_2, \cdots$, *we have*

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

Liu [8] stated that *"Uncertain measure is interpreted as the personal belief degree (not frequency) of an uncertain event that may happen. It depends on the personal knowledge concerning the event. The uncertain measure will change if the state of knowledge changes."* We think that the uncertain measure is a powerful tool in order to analyze an ambiguous phenomena. We will construct Markov decision processes using the uncertain measure in the future.

References

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